Calculus

Lecture 10:

Applications of Derivative: Minimum and Maximum Values

By:

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Overview

- One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems.

- Optimization problems abound in mathematics, physical science and engineering, business and economics, and biology and medicine.

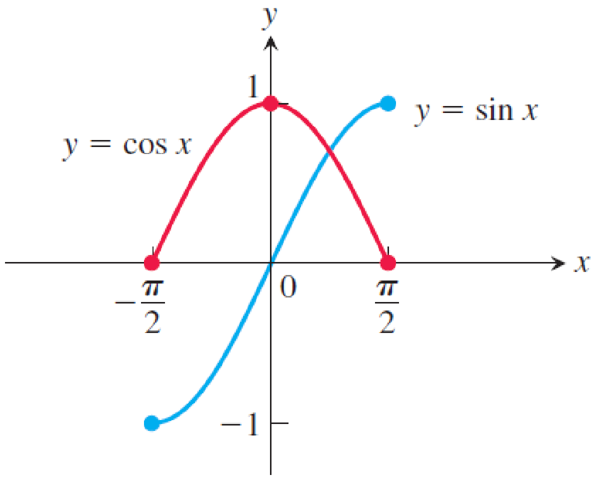
Extreme value of functions

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

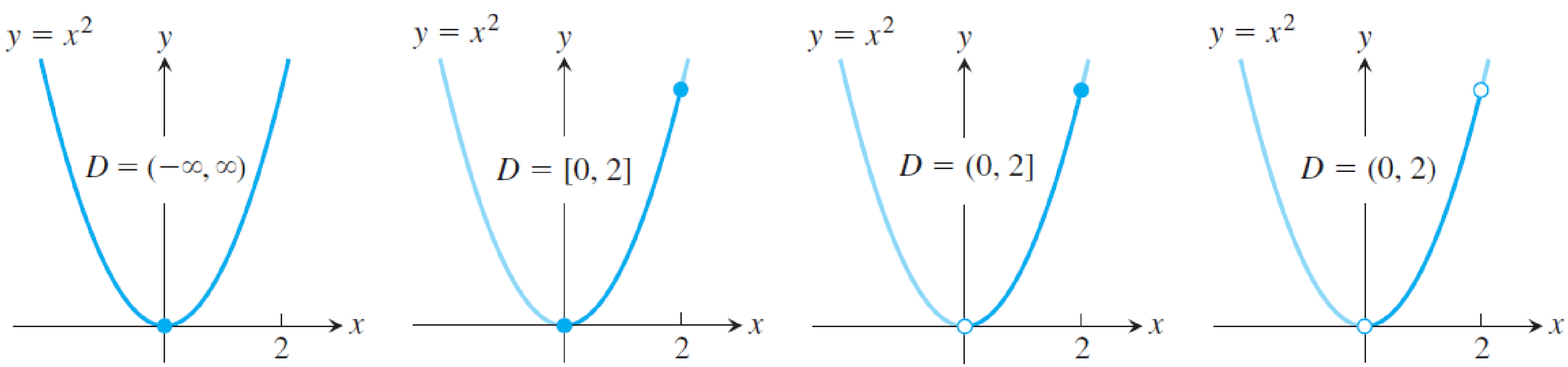
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$



Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.

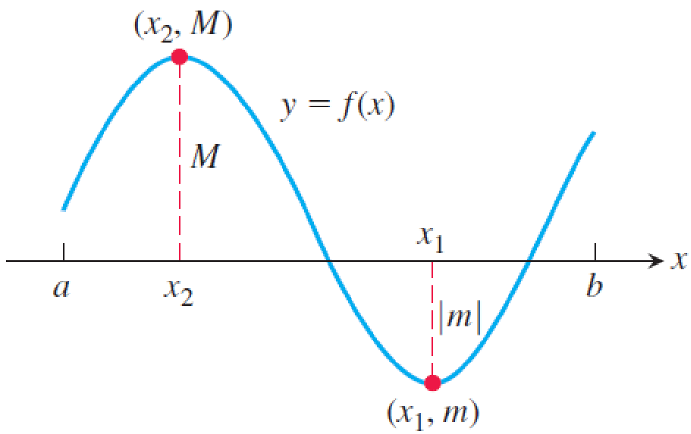
Example

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

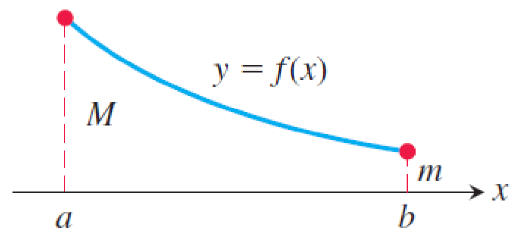


(a) abs min only (b) abs max and min (c) abs max only (d) no max or min

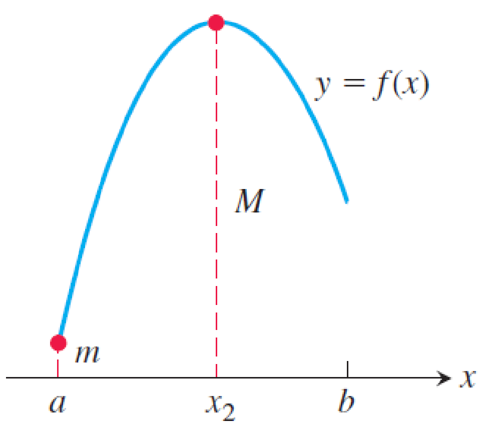
Maximum and minimum points



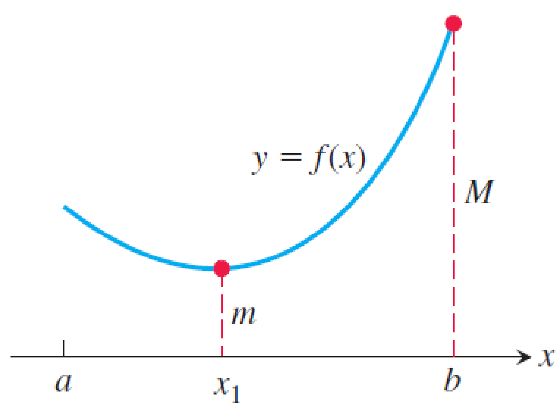
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint

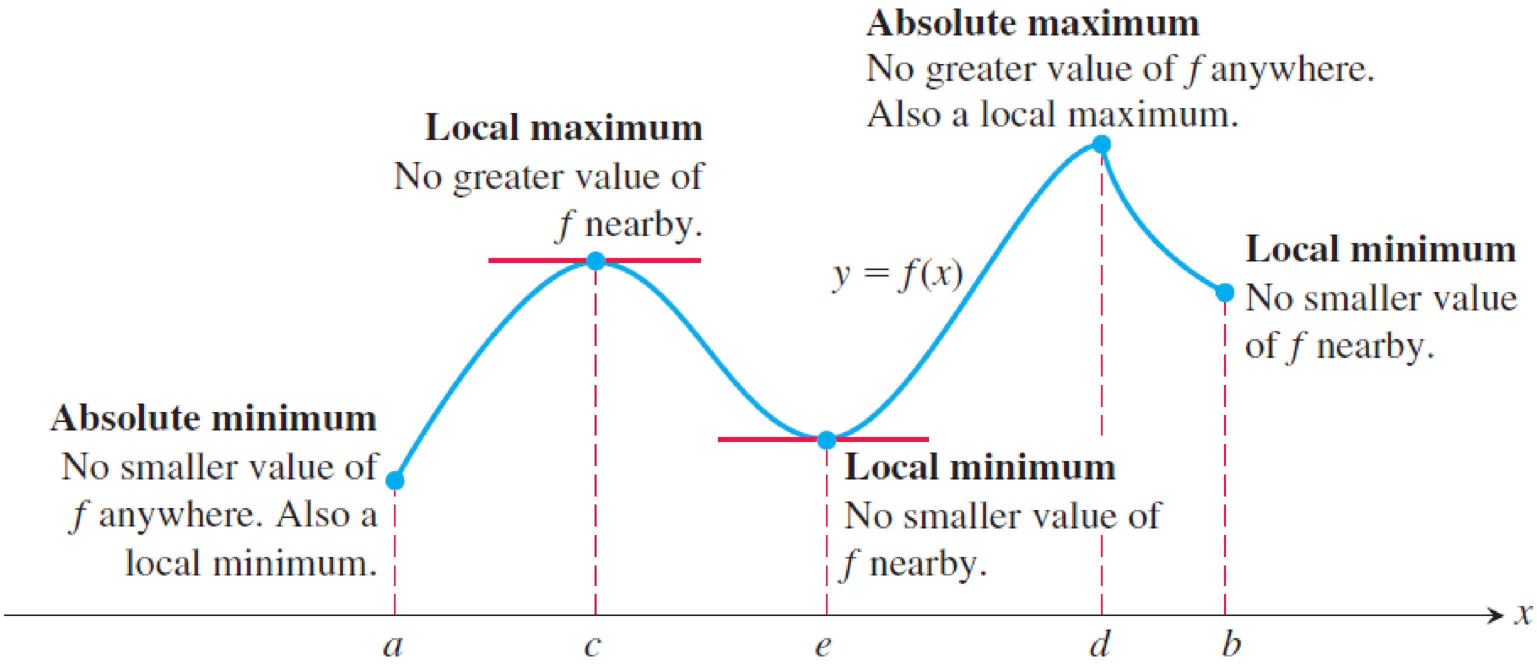


Minimum at interior point, maximum at endpoint

Local extreme values

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

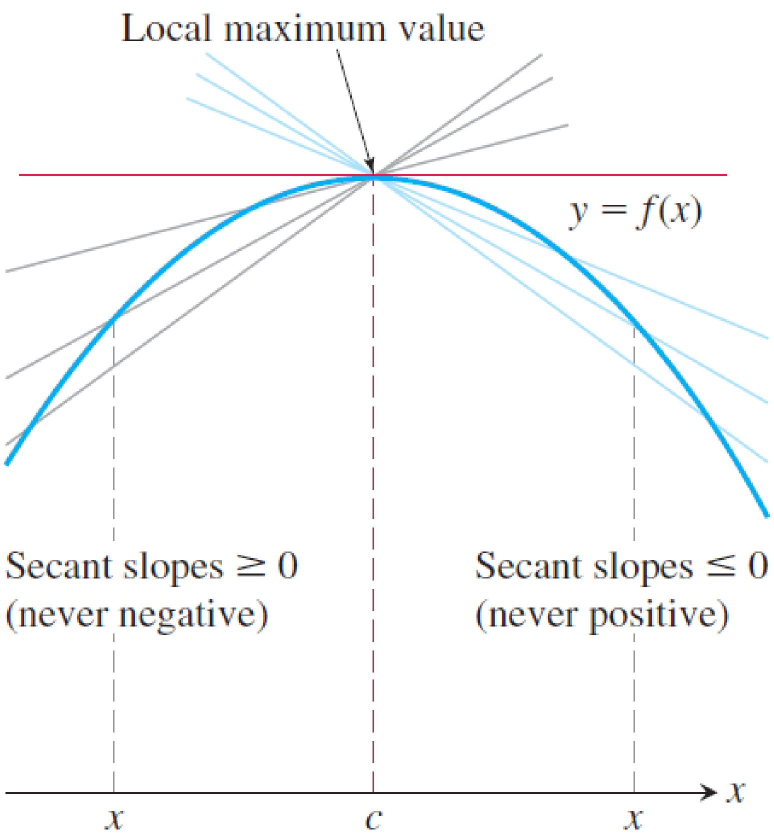
A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.



Finding extreme

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$



Example

$$f(x) = x^4 - 4x^3 + 10$$

First derivative to find extreme values :

$$f'(x) = 0$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

Therefore, the extreme values are : $x = 0$ and $x = 3$

$f'(x) < 0$ | $f'(x) < 0$ | $f'(x) > 0$

f(x) is decreasing | 0 | f(x) is decreasing | 3 | f(x) is increasing

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Second derivative test for local extreme

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example

$$f(x) = x^4 - 4x^3 + 10$$

First derivative (extreme values) :

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

Therefore, the extreme are : $x = 0$ and $x = 3$

$f'(x) < 0$		$f'(x) < 0$		$f'(x) > 0$
f(x) is decreasing	0	f(x) is decreasing	3	f(x) is increasing

Second derivative:

$$f''(x) = 12x^2 - 24x$$

$f''(0) = 0$ ➡ $x = 0$ is neither the minimum point nor the maximum point

$f''(3) = 36$ ➡ $x = 3$ is the minimum point

Example (cont'd)

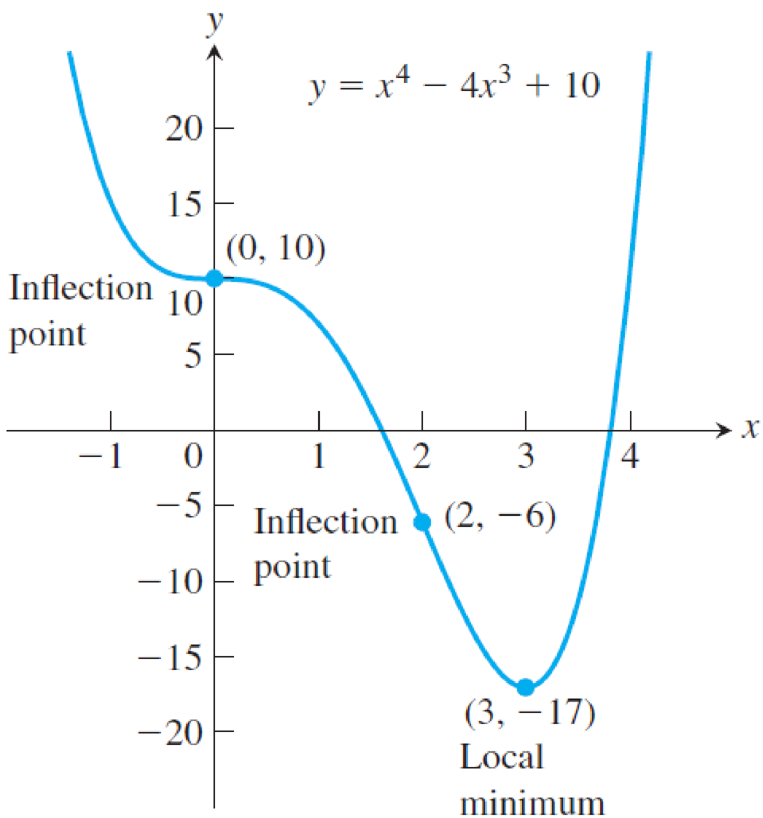
$$f(x) = x^4 - 4x^3 + 10$$

is minimum at x = 3 and the minimum value is:

$$f(3) = 3^4 - 4 \times 3^3 + 10 = -17$$

Therefore, the minimum point is (3,-17).

Example (cont'd)



The minimum point is (3,-17).

$f'(x) < 0$		$f'(x) < 0$		$f'(x) > 0$
f(x) is decreasing	0	f(x) is decreasing	3	f(x) is increasing