

Calculus 1

Lecture 9:

Derivatives of Trigonometric Function and the Chain Rule

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Review

Differentiation Rules (1)

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.9). At every value of x , we find that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0. \quad \blacksquare$$

Differentiation Rules (2)

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \rightarrow x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \quad n \text{ terms} \\ &= nx^{n-1}. \end{aligned}$$



Differentiation Rules (2a)

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x .

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Differentiation Rules (3)

Derivative Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

Proof

$$\begin{aligned} \frac{d}{dx}cu &= \lim_{h \rightarrow 0} \frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition} \\ & && \text{with } f(x) = cu(x) \\ &= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} && \text{Constant Multiple Limit Property} \\ &= c \frac{du}{dx} && u \text{ is differentiable.} \end{aligned}$$

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

Differentiation Rules (4)

Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$


Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}\frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \quad \blacksquare\end{aligned}$$

Example

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$ Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$


Differentiation Rules (5)

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Differentiation Rules (6)

Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

$$(uv)' = uv' + vu'$$

Example

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx} [(x^2 + 1)(x^3 + 3)] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) && \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$\begin{aligned}y &= (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3 \\ \frac{dy}{dx} &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

This is in agreement with our first calculation. ■

Differentiation Rules (7)

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

EXAMPLE 8 Find the derivative of (a) $y = \frac{t^2 - 1}{t^3 + 1}$, (b) $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} & \frac{d}{dt} \left(\frac{u}{v} \right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}. \end{aligned}$$

$$(b) \frac{d}{dx}(e^{-x}) = \frac{d}{dx} \left(\frac{1}{e^x} \right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

Second- and higher- order derivatives

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

y' “y prime”

y'' “y double prime”

$\frac{d^2y}{dx^2}$ “d squared y dx squared”

y''' “y triple prime”

$y^{(n)}$ “y super n”

$\frac{d^n y}{dx^n}$ “d to the n of y by dx to the n”

D^n “D to the n”

Symbols for derivatives

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Derivatives of Trigonometric Function and the Chain Rule

Derivative of the sine function

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule
 $= 2x - \cos x$

(b) $y = e^x \sin x$: $\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \sin x$ Product Rule
 $= e^x \cos x + e^x \sin x$
 $= e^x (\cos x + \sin x)$

(c) $y = \frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$ Quotient Rule
 $= \frac{x \cos x - \sin x}{x^2}$



Derivative of the cos function

$$\frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 2 We find derivatives of the cosine function in combinations with functions.

(a) $y = 5e^x + \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5e^x - \sin x\end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule} \\ &= \sin x(-\sin x) + \cos x(\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

Derivatives of the other basic trigonometric functions (Prove them!)

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

The chain rule

Derivative of composite function

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Examples

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$

Examples (cont'd)

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \\ &= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

Power Chain Rule with
 $u = 5x^3 - x^4, n = 7$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}\left(\frac{1}{3x - 2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\ &= -1(3x - 2)^{-2} \frac{d}{dx}(3x - 2) \\ &= -1(3x - 2)^{-2}(3) \\ &= -\frac{3}{(3x - 2)^2} \end{aligned}$$

Power Chain Rule with
 $u = 3x - 2, n = -1$