Universitas Pembangunan Jaya

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Mata Kuliah : Dinamika Struktur & Pengantar Rekayasa Kegempaan
Kode : TSP – 302
SKS : 3 SKS

Multi Degree of Freedom System

Free Vibration

Pertemuan – 6, 7

- # TIU :
  - Mahasiswa dapat menjelaskan tentang teori dinamika struktur.
  - Mahasiswa dapat membuat model matematik dari masalah teknis yang ada serta mencari solusinya.

- # TIK :
  - Mahasiswa mampu mendefinisikan derajat kebebasan, membangun persamaan gerak sistem MDOF

- Sub Pokok Bahasan :
  - Penentuan derajat kebebasan
  - Properti matrik kekakuan, massa dan redaman

- Structures cannot always be described by a SDoF model
- In fact, structures are *continuous systems* and as such possess an *infinite number of DoF*
- There are analytical methods to describe the dynamic behavior of continuous structures, which are rather complex and require considerable mathematical analysis.
- For practical purposes to study Multi Degree of Freedom System (MDoF), we shall consider the multistory **shear bulding** model.

- A shear building may be defined as a structure in which there is no rotation of a horizontal section at the level of the floors
- Assumption for shear building
  - The total mass of the structure is concentrated at the levels of the floors
  - The slabs/girders on the floors are infinitely rigid as compared to the columns
  - The deformation of the structure is independent of the axial forces present in the columns

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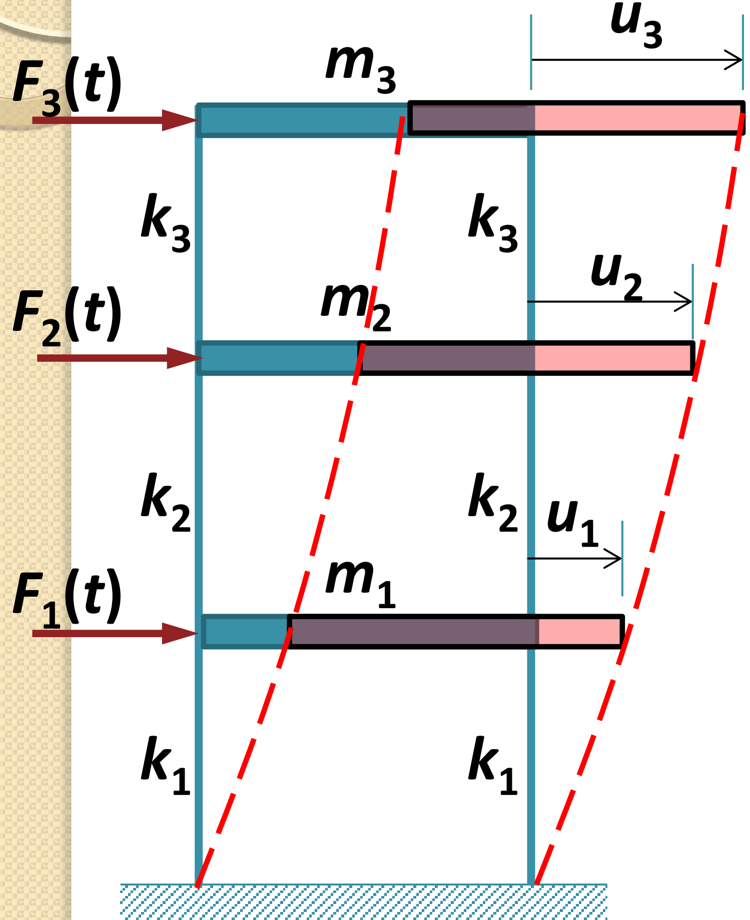
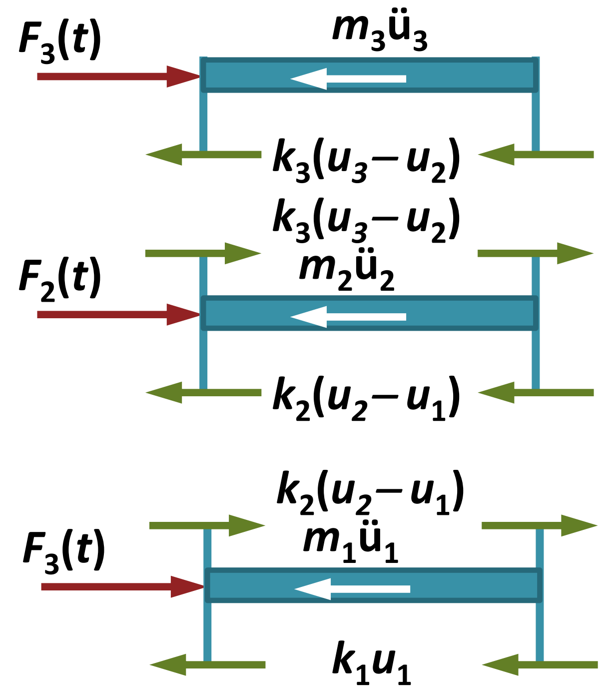


Shear Building w/ 3 DoF



Free Body Diagram

- By equating to zero the sum of the forces acting on each mass :

$$
\begin{aligned}
&m_1\ddot{u}_1 + k_1u_1 - k_2(u_2 - u_1) - F_1(t) = 0 \\
&m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) - F_2(t) = 0 \\
&m_3\ddot{u}_3 + k_3(u_3 - u_2) - F_3(t) = 0
\end{aligned}
\qquad \textbf{(1)}
$$

- Eq. (1) may conveniently be written in matrix notation as :

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\} \qquad \textbf{(2)}$$

- Where :

$$[M]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \qquad [K]=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

- And :

$$\{u\}=\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \{\ddot{u}\}=\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} \qquad \{F\}=\begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}$$

Natural Frequencies and Normal Modes

- EoM for MDoF system in free vibration is :

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\} \quad \textbf{(3)}$$

- Solution of Eq. (3) is :

$$\{u\}=\{\phi_n\}q_n(t)=\{\phi_n\}(A_n\, cos\,\omega_n t+B_n\, sin\,\omega_n t) \quad \textbf{(4)}$$

- The substitution of Eq. (4) into Eq. (3) gives :

$$\left[[K]\{\phi_n\}-\omega_n{}^2[M]\{\phi_n\}\right]q_n(t)=\{0\} \quad \textbf{(5)}$$

- The formulation of Eq (5) is known as an eigenvalue problem.
- To indicate the formal solution to Eq. (5), it is rewritten as :

$$\left[[K]-\omega_n^{\,2}[M]\right]\{\phi_n\}=\{0\} \qquad \textbf{(6)}$$

- Eq (6) has non trivial solution if :

$$\left|[\mathrm{K}]-\omega_n^{\,2}[M]\right|=0 \qquad \textbf{(7)}$$

- Eq.(7) gives a polynomial equation of degree n in ω_n^2, known as *characteristic equation of the system*

A vibrating system with n DoF has n natural vibration frequencies ω_n, arranged in sequence from smallest to largest

- The n roots of Eq. (7) determine the n natural frequencies ω_n of vibration.
- These roots of the characteristic equation are also known as **eigenvalues**
- When a natural frequency ω_n is known, Eq.(6) can be solved for the corresponding vector ϕ_n
- There are n independent vectors ϕ, which are known as natural modes of vibration, or natural mode shape of vibration
- These vectors are also known as **eigenvector**.

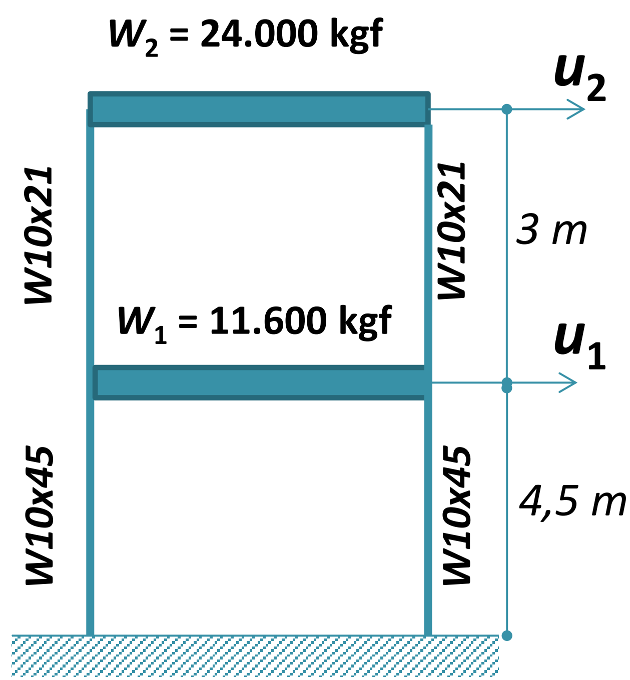
Example 1

- Determine :
  1. The natural frequencies and corresponding modal shapes

W10x21 → I = 4,900 cm⁴

W10x45 → I = 10,300 cm⁴

E = 200.000 MPa



$\omega_1 = 10{,}76 \qquad \omega_2 = 38{,}28$

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 0{,}6809 & -3{,}0385 \\ 1 & 1 \end{bmatrix}$$

Modal & Spectral Matrices

- The N eigenvalues, N natural frequencies, and n natural modes can be assembled compactly into matrices

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

DoF 1, 2, ..., n

Mode 1, 2, n

→ Modal matrix

$$[\Omega^2] = \begin{bmatrix} {\omega_1}^2 & & & \\ & {\omega_2}^2 & & \\ & & \ddots & \\ & & & {\omega_n}^2 \end{bmatrix}$$

→ Spectral matrix

Orthogonality of Modes

- The natural modes corresponding to different natural frequencies can be shown to satisfy the following orthogonality conditions :

$$\left.\begin{aligned}\{\phi_n\}^T[K]\{\phi_r\}=0\\ \{\phi_n\}^T[M]\{\phi_r\}=0\end{aligned}\right\} \quad \text{When } \omega_n \neq \omega_r$$

Free Vibration Response : Undamped System

- EoM for MDoF Undamped Free Vibration is :

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\} \qquad \textbf{(8)}$$

- The differential equation (8) to be solved had led to the matrix eigenvalue problem of Eq.(6).
- The general solution of Eq. (8) is given by a superposition of the response in individual modes given by Eq. (4).

$$\{u\}=\{\phi_1\}q_1+\{\phi_2\}q_2+\{\phi_3\}q_3+\ldots+\{\phi_n\}q_n \qquad \textbf{(9)}$$

- Or $\{u\}=\sum_{n=1}^{N}\phi_n q_n$ $\qquad$ $\{u\}=[\phi]\{q\}$ **(10)**

- Premultiply Eq. (10) with $\{\phi_n\}^T[M]$:

$$\{\phi_n\}^T[M]\{u\} = \{\phi_n\}^T[M][\phi]\{q_n\} \qquad \textbf{(11)}$$

$$= \{\phi_n\}^T[M]\{\phi_1\}q_1 + \{\phi_n\}^T[M]\{\phi_2\}q_2 + \ldots.$$

$$+ \{\phi_n\}^T[M]\{\phi_N\}q_N$$

- Regarding the orthogonality conditions :

$$\{\phi_n\}^T[M]\{u\} = \{\phi_n\}^T[M]\{\phi_n\}q_n$$

- From which :

$$q_n = \frac{\{\phi_n\}^T[M]\{u\}}{\{\phi_n\}^T[M]\{\phi_n\}} \qquad \text{also} \qquad \dot{q}_n = \frac{\{\phi_n\}^T[M]\{\dot{u}\}}{\{\phi_n\}^T[M]\{\phi_n\}} \qquad \textbf{(11)}$$

- If Eq. (8) is premultiplied by the transpose of the nth mode-shape vector $\{\phi_n\}^T$, and using Eq.(10) with its second derivative it becomes :

$$\underbrace{\{\phi_n\}^T[M]\{\phi_n\}}_{M_n}\ddot{q}_n + \underbrace{\{\phi_n\}^T[K]\{\phi_n\}}_{K_n}q_n = 0 \quad \textbf{(11)}$$

Generalized mass — Generalized stiffness

- Eq.(11) is an EoM from SDoF Undamped System for mode n, which has solution :

$$q_n(t) = q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t \quad \textbf{(12)}$$

- Solution for Eq. (8) becomes :

$$\{u\} = \sum_{n=1}^{N} \phi_n \left[q_n(0) cos\, \omega_n t + \frac{\dot{q}_n(0)}{\omega_n} sin\, \omega_n t \right] \qquad \textbf{(13)}$$

- The procedure described above can be used to obtain an independent SDoF equation for each mode of vibration of the undamped structure.
- This procedure is called the **mode-superposition method**

Example 2

- Based on data from Example 1, and the following initial condition, for each DoF plot the time history of displacement, regarding the 1st, and 2nd mode contribution.

$$u_{(t=0)} = \begin{Bmatrix} 6 \cdot 10^{-3} \\ 8 \cdot 10^{-3} \\ 10 \cdot 10^{-3} \end{Bmatrix} m \qquad \dot{u}_{(t=0)} = \begin{Bmatrix} 0 \\ 0{,}20 \\ 0 \end{Bmatrix} \mathrm{m/s}$$

Free Vibration Response : Damped System

- EoM for MDoF Damped Free Vibration is :

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{0\} \qquad \textbf{(14)}$$

- Using Eq. (10) and its derivative :

$$[M][\phi]\{\ddot{q}\}+[C][\phi]\{\dot{q}\}+[K][\phi]\{q\}=\{0\} \qquad \textbf{(15)}$$

- Premultiply Eq. (15) with $\{\phi\}^T$:

$$\underbrace{[\phi]^T[M][\phi]}_{M_n}\{\ddot{q}\}+\underbrace{[\phi]^T[C][\phi]}_{C_n}\{\dot{q}\}+\underbrace{[\phi]^T[K][\phi]}_{K_n}\{q\}=\{0\} \qquad \textbf{(16)}$$

- Eq.(16) is an EoM from SDoF Damped System for mode n, which has solution :

$$q_n(t) = e^{-\xi_n \omega_n t}\left[q_n(0) \cos \omega_{nD} t + \frac{\dot{q}_n(0) + \xi_n \omega_n q_n(0)}{\omega_{nD}} \sin \omega_{nD} t \right] \qquad \textbf{(17)}$$

- The displacement response of the system is then obtained by substituting Eq. (17) in Eq. (10)

$$u(t) = \sum_{n=1}^{N} \phi_n e^{-\xi_n \omega_n t}\left[q_n(0) \cos \omega_{nD} t + \frac{\dot{q}_n(0) + \xi_n \omega_n q_n(0)}{\omega_{nD}} \sin \omega_{nD} t \right] \qquad \textbf{(18)}$$