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Mata Kuliah : Dinamika Struktur & Pengantar Rekayasa Kegempaan
Kode : TSP – 302
SKS : 3 SKS

Single Degree of Freedom System

Free Vibration

Pertemuan - 2

- TIU :
  - Mahasiswa dapat menjelaskan tentang teori dinamika struktur.
  - Mahasiswa dapat membuat model matematik dari masalah teknis yang ada serta mencari solusinya.

- TIK :
  - Mahasiswa dapat memformulasikan persamaan gerak sistem struktur berderajat kebebasan tunggal yang bergetar bebas

- Sub Pokok Bahasan :
  - Getaran Bebas SDoF Tak Teredam
  - Frekuensi dan Periode
  - Getaran Bebas SDoF Dengan Redaman

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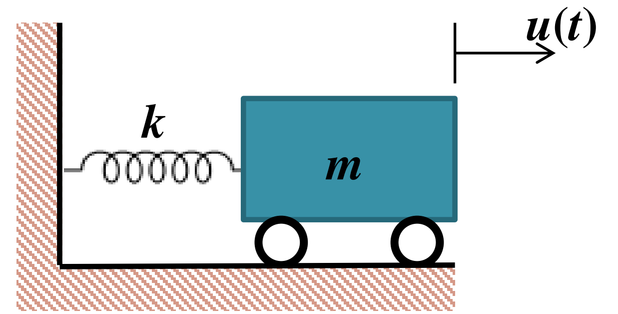
Undamped SDoF Free-Vibration

- A structure is said to be undergoing free vibration when it is disturbed from its static equilibrium position and then allowed to vibrate **without** any external dynamic excitation.



- The differential equation governing free vibration of the systems without damping ($c = 0$ & $p(t) = 0$) is :

$$\boxed{m\ddot{u}(t) + ku(t) = 0} \quad \textbf{(1)}$$

- This equation called *second order homogeneous differential equation*

- To find the solution of Eq. (1), assuming a trial solution given by :

$$u = A\cos\omega_n t \qquad \textbf{(2.a)} \qquad\qquad u = B\sin\omega_n t \qquad \textbf{(2.b)}$$

- Where A and B are constants depending on the initiation of the motion while ω is a quantity denoting a physical characteristic of the system.

- The substitution eq.(2.a&b) into eq. (1) gives :

$$\left(-m\omega_n^{\ 2}+k\right)A\cos\omega_n t=0 \qquad \textbf{(3)}$$

- If eq.(3) is to be satisfied at any time, the factor in parentheses must be equal to zero, or :

$$\omega_n^{\ 2}=\frac{k}{m} \Rightarrow \boxed{\omega_n=\sqrt{\frac{k}{m}}} \qquad \textbf{(4)}$$

- ω_n is known as the natural circular frequency of the system (rad/s)

- Since either eq.(2.a) or (2.b) is a solution of eq.(1), and since the differential equation is linear, the superposition of these two solution is also solution :

$$u = A\cos\omega_n t + B\sin\omega_n t \qquad \textbf{(5)}$$

- The expression for velocity, *ú*, is :

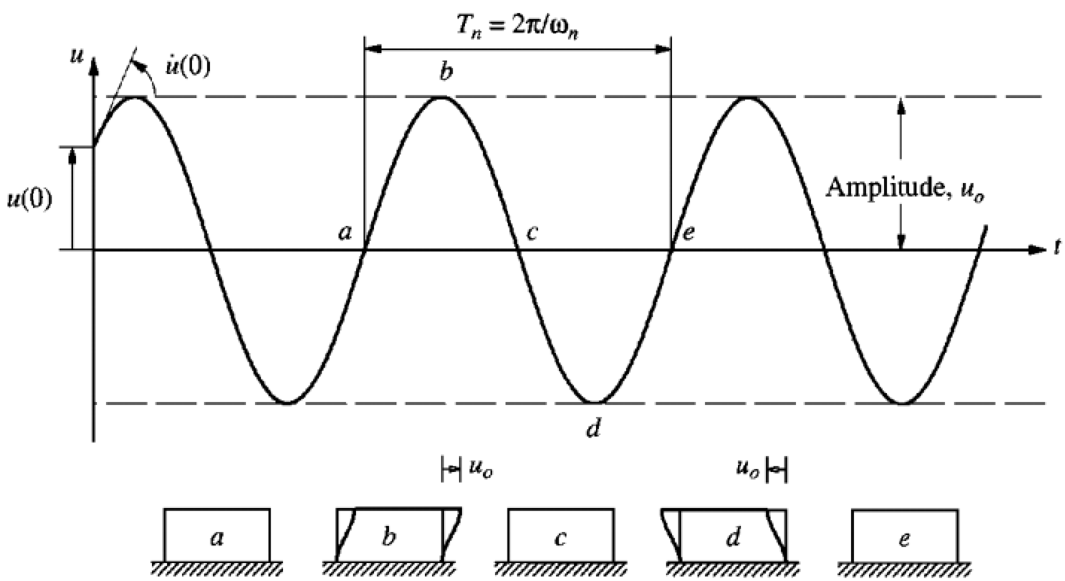
$$\dot{u} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t \qquad \textbf{(6)}$$

- Constant A and B, will be determined based on initial condition (at t = 0) :

$$u = u(0) \qquad \dot{u} = \dot{u}(0)$$

- Subject to these initial conditions, the solution to the eq. (5) will be :

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t \qquad \textbf{(7)}$$



$$u_0 = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0)}{\omega_n}\right]^2}$$

Frequency and Period

- The time required for the undamped system to complete one cycle of free vibration is the ***natural period of vibration***, T_n (in second)

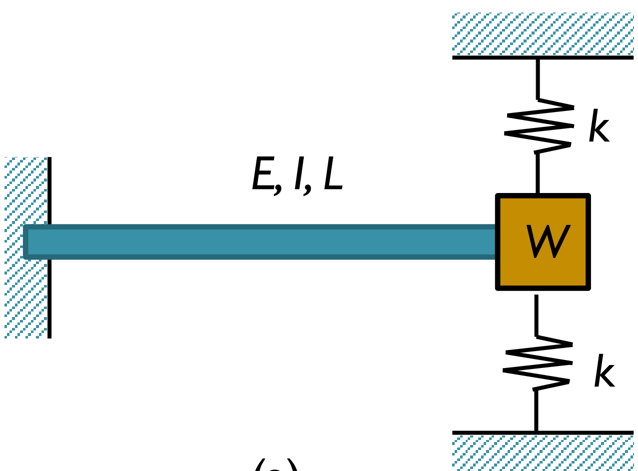
$$T_n = \frac{2\pi}{\omega_n} \qquad \textbf{(8)}$$

- A system executes $1/T_n$ cycles in 1 sec. This ***natural cyclic frequency*** of vibration (in Hz) is denoted by :

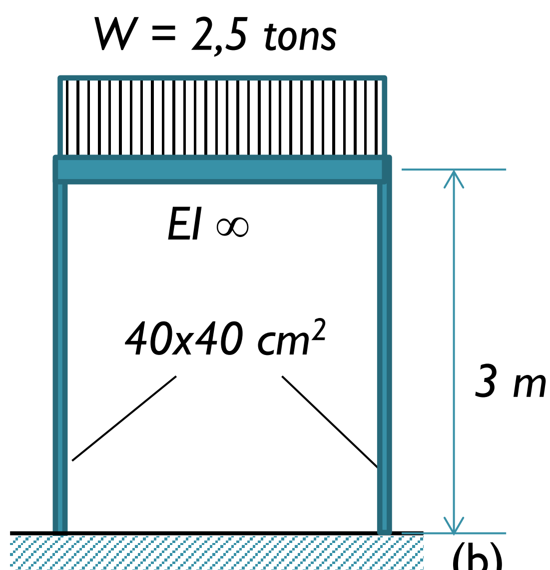
$$f_n = \frac{\omega_n}{2\pi} \qquad \textbf{(9)}$$

Exercise

- Determine natural frequency and natural period from each system below



(a)

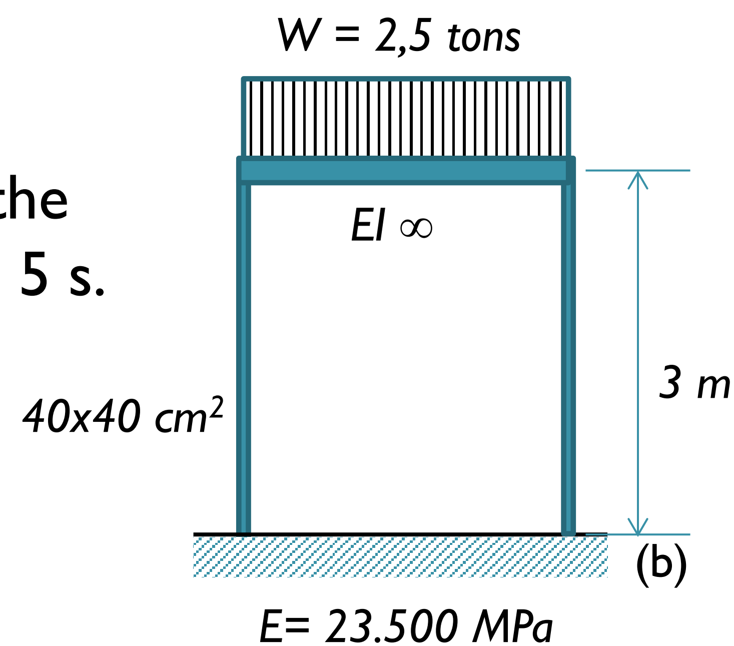


(b)

E= 23.500 MPa

Assignment 1

- If system (b) have initial condition $u(0) = 3\,\text{cm}\ \&\ \dot{u}(0) = 20\,\text{cm/s}$
  a) Determine the displacement at t = 2 sec
  b) Plot a time history of displacement response of the system, for t = 0 s until t = 5 s.



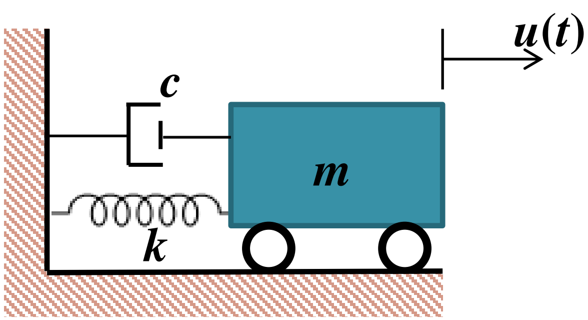
Damped SDoF Free-Vibration

- If damping is present in the system, the differential equation governing free vibration of the systems is :

$$\boxed{m\ddot{u}(t)+c\dot{u}(t)+ku(t)=0} \qquad \textbf{(10)}$$

- The trial solution to satisfy Eq. (14) is $u = Ce^{pt}$ **(11)**
- Subtitute Eq. (11) to Eq. (10) yield the characteristic eq. :

$$mp^2 + cp + k = 0 \qquad \textbf{(12)}$$



- The roots of the quadratic Eq. (12) are :

$$p_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad \textbf{(13)}$$

- Thus the general solution of Eq. (10) is :

$$u(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} \qquad \textbf{(14)}$$

- Three distinct cases may occur depends on the sign under the radical in Eq. (13)

1. Critically Damped System

- For critical damping :

$$\boxed{c = c_{cr} = 2m\omega_n = 2\sqrt{km}} \qquad \textbf{(15)}$$

- The expression under the radical in Eq. (13) is equal to zero, and the roots are equal, they are :

$$p_1 = p_2 = -\frac{c_{cr}}{2m} = -\omega_n \qquad \textbf{(16)}$$

- The general solution of Eq. (14) is :

$$u(t) = (A + Bt)e^{-\omega_n t} \qquad \textbf{(17)}$$

1. Critically Damped System

- Constant A and B, will be determined based on initial condition (at t = 0) :

$$u = u(0) \qquad \dot{u} = \dot{u}(0)$$

- Substitute the initial condition, yield :

$$\boxed{u(t) = e^{-\omega_n t}\left[u(0) + \left(\dot{u}(0) + \omega_n u(0)\right)t\right]} \qquad \textbf{(18)}$$

2. Overdamped System

- In an overdamped system, the damping coefficient is greater that the value for critical damping ($c > c_{cr}$)
- EoM of an overdamped system due to initial condition is :

$$u(t) = e^{-\xi\omega_n t}\left[Ae^{-\omega'_D t} + Be^{\omega'_D t}\right] \qquad \textbf{(19)}$$

- Where :

$$A = \frac{-\dot{u}(0) + \left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n u(0)}{2\omega'_D} \qquad \omega'_D = \omega_n\sqrt{\xi^2 - 1}$$

$$B = \frac{\dot{u}(0) + \left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n u(0)}{2\omega'_D} \qquad \xi = \frac{c}{c_r}$$

- ξ is the **damping ratio** (%)

3. Underdamped System

- When the value of the damping coefficient is less than the critical value ($c < c_{cr}$), the roots of the characteristic eq. (12) are complex conjugates, so that :

$$p_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad i = \sqrt{-1}$$

- Regarding the Euler's equations which relate exponential and trigonometric functions :

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

3. Underdamped System

- The substitution of the roots p_1 and p_2 into Eq. (14) together with the use of Euler equation, gives :

$$u(t) = e^{-\xi\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) \tag{20}$$

- Where $\omega_D = \omega\sqrt{1-\xi^2}$
- Using initial condition, $u = u(0)$ $\dot{u} = \dot{u}(0)$

$$\boxed{u(t) = e^{-\xi\omega_n t}\left(u(0)\cos\omega_D t + \frac{\dot{u}(0) + \xi\omega_n u(0)}{\omega_D}\sin\omega_D t\right)} \tag{21}$$

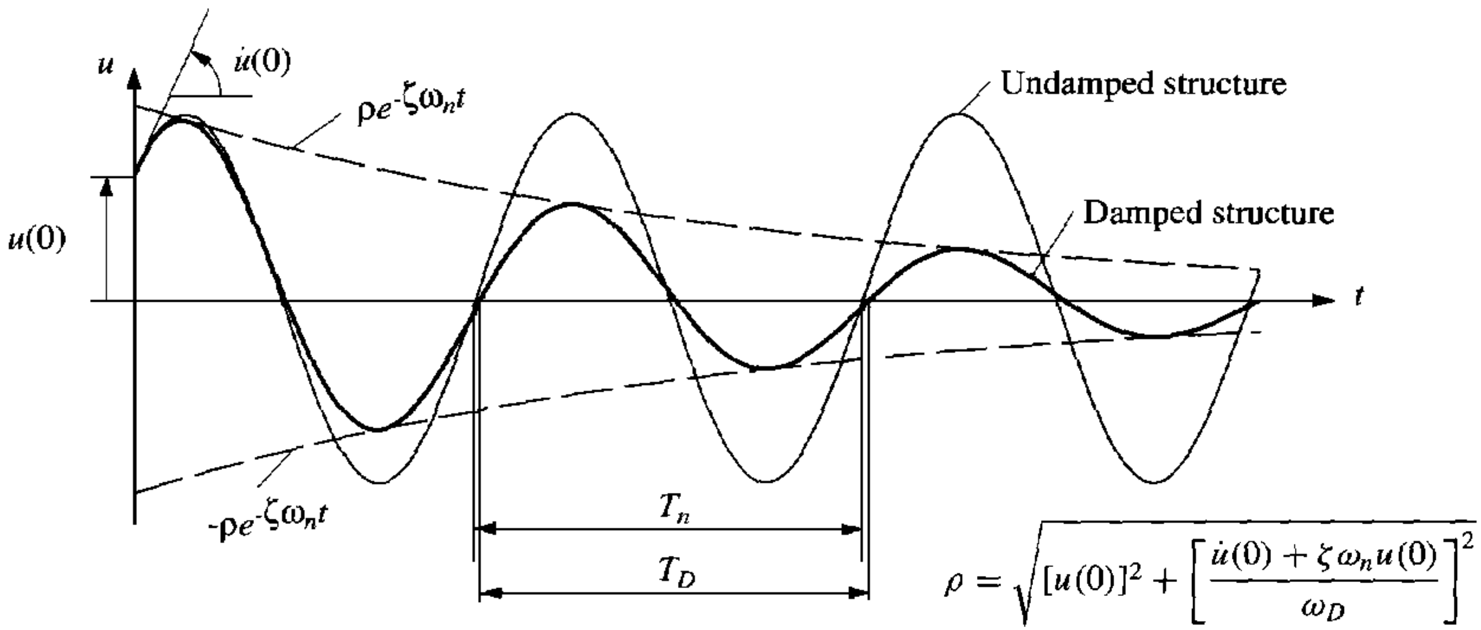


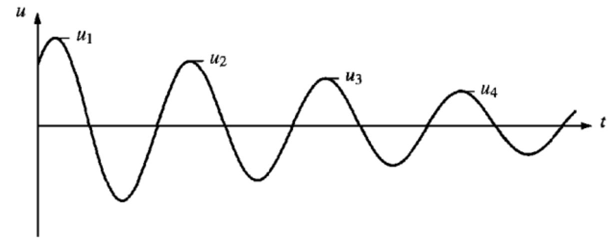
Figure 2.2.2 Effects of damping on free vibration.

Many types of structures, such as buildings, bridges, dams, nuclear power plants, offshore structures, etc., all fall into underdamped system ($c < c_{cr}$), because typically their damping ratio is less than 0,10.

Logarithmic Decrement

- The ratio u_i/u_{i+1} of successive peaks (maxima) is :

$$\frac{u_i}{u_{i+1}} = exp\left(\frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}}\right) \qquad \textbf{(22)}$$



- The natural logarithm of this ratio, called the logarithmic decrement,

$$\delta = \ln\frac{u_i}{u_{i+1}} = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} \qquad \textbf{(23)}$$

- If ζ is small, this gives an approximate equation : $\delta \approx 2\pi\zeta$

- Over j cycles the motion decreases from u_1 to u_{j+1}. This ratio is given by :

$$\frac{u_1}{u_{j+1}} = \frac{u_1}{u_2} \cdot \frac{u_2}{u_3} \cdot \frac{u_3}{u_4} \dots \frac{u_j}{u_{j+1}} = e^{j\delta} \qquad \textbf{(24)}$$

- Therefore,

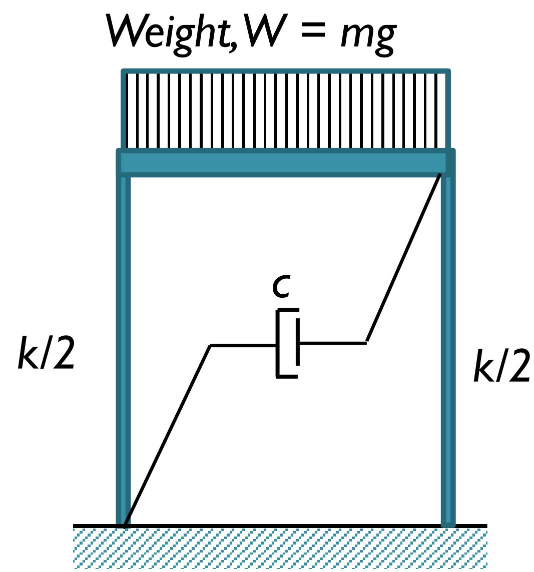
$$\delta = \frac{1}{j} ln \frac{u_1}{u_{j+1}} \cong 2\pi\xi \qquad \textbf{(25)}$$

Exercise

A one-story building is idealized as a rigid girder supported by weightless columns. In order to evaluate the dynamic properties of this structure, a free-vibration test is made, in which the roof system (rigid girder) is displaced laterally by a hydraulic jack and then suddenly released.

During the jacking operation, it is observed that a force of 10 kg is required to displace the girder 0.508 cm.

After the instantaneous release of this initial displacement, the maximum displacement on the first return swing is only 0.406 cm and the period of this displacement cycle is T = 1.40 sec.



Find :

- W
- f and ω
- δ, ξ, c, ω_D
- u_6

Assignment 2

- If system in the figure have initial condition $u(0) = 3\,\text{cm}\ \&\ \dot{u}(0) = 20\,\text{cm/s}$
- Plot a time history of displacement response of the system, for t = 0 s until t = 5 s, if :
  a. $\xi = 5\%$
  b. $\xi = 100\%$
  c. $\xi = 125\%$